

SUPER-EXTREME EVENT'S INFLUENCE ON A WEIERSTRASS-MANDELBROT CONTINUOUS-TIME RANDOM WALK

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Two utmost cases of *super-extreme event's* influence on the velocity autocorrelation function (VAF) were considered. The VAF itself was derived within the hierarchical Weierstrass-Mandelbrot Continuous-Time Random Walk (WM-CTRW) formalism, which is able to cover a broad spectrum of continuous-time random walks. Firstly, we studied a super-extreme event in a form of a sustained drift, whose duration time is much longer than that of any other event. Secondly, we considered a super-extreme event in the form of a shock with the size and velocity much larger than those corresponding to any other event. We found that the appearance of these super-extreme events substantially changes the results determined by extreme events (the so called "black swans") that are endogenous to the WM-CTRW process. For example, changes of the VAF in the latter case are in the form of some instability and distinctly differ from those caused in the former case. In each case these changes are quite different compared to the situation without super-extreme events suggesting the possibility to detect them in natural system if they occur.

PACS numbers: 89.65.Gh, 02.50.Ey, 02.50.Ga, 05.40.Fb, 02.30.Mv

I. INTRODUCTION

One of the most remarkably emerging observation within the natural and socio-economical sciences is that the empirical data which they supply are frequently punctuated by rare extreme events or *black swans* which can play a dominant role. This observation is usually quantified by power-laws or heavy-tailed probability distributions of event sizes (cf. [1–13] and references therein). However, as it was pointed out in [13], there is an empirical evidence that something important beyond power-laws does exist. In this context, the concept of *super-extreme events*, *outliers* or *dragon-kings* was introduced.

By the super-extreme event, outlier or dragon-king¹ we mean an event of the values that are abnormally different than values of other events in a random sample taken from a given population [15]. It is therefore an anomaly, an event usually removed in order to obtain reliable statistical estimations. The term "outlier" emphasizes the spurious nature of these anomalous events, suggesting discarding them as errors or misleading monsters. In contrast, the term "dragon-king" emphasizes their relevance in the dynamics and the importance of keeping them to be able to understand the generating process. The super-extreme events are statistically complementary to extreme events, as documented in [13]. Moreover, the idea that dragon-kings are often associated with the occurrence of a catastrophe, a phase transition, and bifurcation as well as with a tipping point, whose emergent organization produces visible

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¹ The poetic term "dragon-king" stresses that: (i) we deal with exceptional event which is a completely different kind of "animal" in comparison with the usual ones concerning the rest of events, (ii) it emphasizes the importance of this event being outside the power-law like an absolute monarchy standing above the law or as the wealth of a monarch owning a finite part of the whole country, beyond the Pareto distribution of its citizen's wealths [14].

precursors was also developed there.

The main goal of the present work is to analytically demonstrate and numerically simulate two utmost cases of the influence of dragon-kings on the velocity autocorrelation function (VAF) of a random walker. Herein, we studied VAF in the frame of the Weierstrass-Mandelbrot Continuous-Time Random Walk (WM-CTRW) formalism developed in [16–18]. This model is a hierarchical version of the canonical Continuous-Time Random Walk (CTRW) formalism [19–21] in which the hierarchical spatio-temporal waiting-time distribution (WTD) was assumed as its basic quantity (cf. Section III A in this work as well as Equation (20) in [17]).

Note that the WM-CTRW formalism is sufficiently generic and flexible. It is able to cover various types of diffusion, i.e. from normal diffusion, through the superdiffusion (e.g. the persistent fractional Brownian motion, fBm, [22]), and the ballistic one, to the Lévy walk. Remarkably, the extreme events are contained in the spatio-temporal structure of time series obtained within the WM-CTRW formalism by stochastic simulation. This is explained in details in Section II. The extreme events moderate the relaxation of the system to equilibrium (or partial equilibrium), e.g. by changing the relaxation from exponential to power-law. Furthermore, we used the WM-CTRW formalism because we (superficially) verified that the hierarchical WTD described quite well its empirical counterparts obtained for continuous quotation of the exchange rates on a currency exchange market as well as of the share price trading on a stock exchange.

The present paper is organized in the following manner. First (in Section II), our problem is defined along with definitions of the most relevant quantities used in our study. Then (Section III), the WM-CTRW formalism is defined and, next, a superdiffusion phase is discussed. Subsequently, Section IV presents derivation of VAF including a dragon-king event. The comparison of predictions of our theoretical formulae for the VAF with the results of the simulations is contained in Section V. Finally, summary and concluding remarks are presented in Section VI.

II. DEFINITION OF THE PROBLEM

In the present work we consider, as an example, a superdiffusion case within the WM-CTRW formalism, where so-called weak ergodicity is obeyed [23–25]; that is, the mean value of the waiting-times between turning points of a random walk trajectory is finite. As the probability of appearance of a dragon-king is extremely small (that is, the waiting time for its appearance is too long for all practical purposes), it was produced after the simulation and next put "manually" somewhere inside the series. In that sense, the dragon-king's appearance can be considered as an exogenous event.

We answer the question of how much the stationary velocity autocorrelation function, $C(\Delta t)$, derived from a given time series is changed when this time series is suddenly punctuated by a single-step super-extreme event. We consider two types of super-extreme events:

- (i) the long-drawn event which has super-extremely long duration time t_d (cf. Figure 1) and
- (ii) the shock, or sudden jump, of a random variable X , which has super-extreme velocity v_d (cf. Figure 2).

The background of the corresponding dragon-kings' definition is again presented in Figures 1 and 2. The resulting VAF, involving a dragon-king, is denoted below by $C_d(\Delta t)$.

The VAF is defined as

$$\begin{aligned} \text{VAF}(\Delta t) &= \langle v(t') v(t' + \Delta t) \rangle - \langle v(t') \rangle \langle v(t' + \Delta t) \rangle \\ &= \langle v_1 v_2 \rangle (\Delta t) - \langle v_1 \rangle \langle v_2 \rangle = \begin{cases} C(\Delta t), & \text{in the absence of a dragon-king,} \\ C_d(\Delta t), & \text{in the presence of a dragon-king,} \end{cases} \end{aligned} \quad (1)$$

where $\langle \dots \rangle$ means the moving average (or averaging over current time $t' \leq t_{tot} - \Delta t$) in a given time-window Δt (cf. Figure 1 and 2 and considerations in Section IV) and $v(t') \stackrel{\text{def}}{=} [X(t') - X(t' - dt)]/dt$, where the time-discretization step is $dt \leq \Delta t$, t' . The velocities v_1 and v_2 are defined here at the beginning and at the end of the time-window Δt , respectively.

Both quantities, $C(\Delta t)$ and $C_d(\Delta t)$, are studied analytically (in Sections III and IV) and by numerical simulations (in Section V) because our task is to find relations between $C(\Delta t)$ and $C_d(\Delta t)$ and verify them by simulations for cases (i) and (ii) mentioned above.

III. WEIERSTRASS-MANDELBROT CONTINUOUS-TIME RANDOM WALK

The task of this section is to briefly sketch elements of the WM-CTRW formalism that are useful for our analysis. An explicit form of the VAF, $C(\Delta t)$, in the absence of a dragon-king was described in details in our earlier works

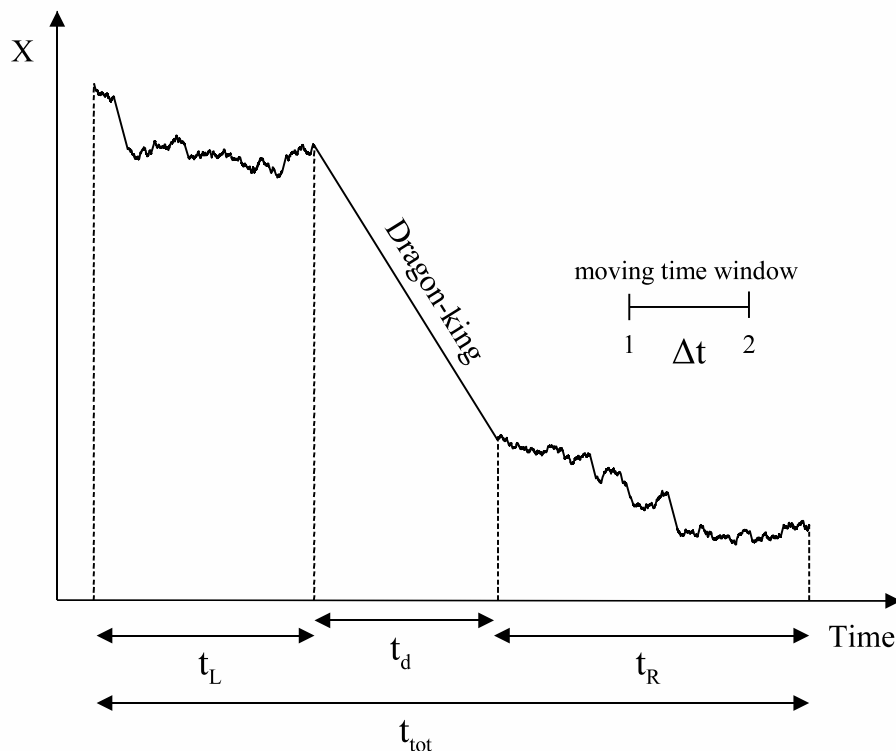


FIG. 1: Schematic time series for the total time t_{tot} containing the dragon-king event (represented by the longest sloping straight line) having constant velocity v_d and duration time t_d . The time series before and after the occurrence of the dragon-king have duration times t_L and t_R , respectively. Obviously, $t_{tot} = t_L + t_d + t_R$.

[17, 18].

A. Definition of the WM-CTRW formalism

The Weierstrass-Mandelbrot Continuous-Time Random Walk is defined by the hierarchical spatio-temporal waiting-time distribution (WTD). This WTD² is given by the following weighted hierarchical series

$$\psi(x, t) = \sum_{j=0}^{\infty} w(j) \psi_j(x, t), \quad (2)$$

where x is the walker single-step spatial displacement passed (with constant velocity) within the time interval t and the weight $w(j)$ is given by the probability distribution

$$w(j) = \frac{1}{N^j} \left(1 - \frac{1}{N} \right), \quad N > 1, \quad j = 0, 1, 2, \dots \quad (3)$$

This weight can be interpreted as the probability of exactly j consecutive successes in some Bernoulli series, where $1/N$ is the probability of a single success and the conditional single-level WTD is assumed in the factorized form of two different single-variable distributions f and h

$$\psi_j(x, t) = \frac{1}{v_0 v^j t} f\left(\frac{|x|}{v_0 v^j t}\right) \frac{1}{\tau_0 \tau^j} h\left(\frac{t}{\tau_0 \tau^j}\right). \quad (4)$$

² The complete definition of the WM-CTRW model additionally requires a special treatment of the first step of the random walker [17]. However, it is irrelevant when moving-average is performed. Therefore, we do not consider this special treatment in this work.

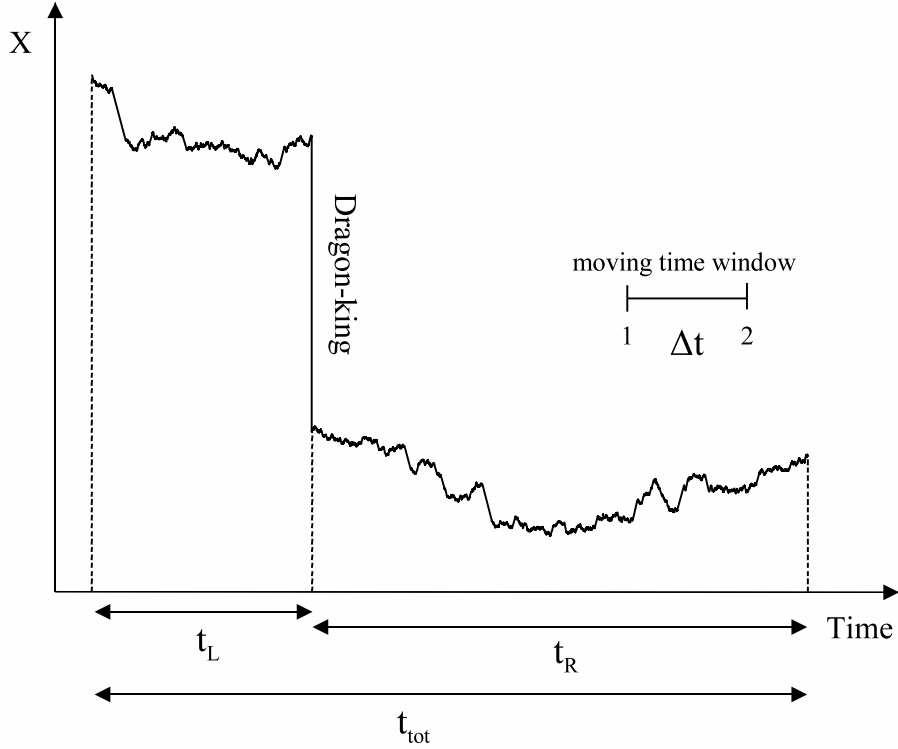


FIG. 2: Schematic time series containing a dragon-king in the form of a shock (represented by the longest vertical straight line). The time series before and after the dragon-king have duration times t_L and t_R , respectively while the duration time of the shock is $t_d = dt$ being so short that it cannot be visualized in the plot ($t_{tot} = t_L + dt + t_R$).

In Expression (4), we used a simple representation of a random walk (but not random jumps or flights). That is, conditional temporal and spatial probability distributions we assumed are of the forms

$$f\left(\frac{|x|}{v_0 v^j t}\right) = \frac{1}{2} \delta\left(\frac{|x|}{v_0 v^j t} - 1\right) \quad (5)$$

and

$$h\left(\frac{t}{\tau_0 \tau^j}\right) = \exp\left(-\frac{t}{\tau_0 \tau^j}\right), \quad (6)$$

respectively³. The mean duration time, $\tau_0 \tau^j$, of the random walker single step and its velocity, $v_0 v^j$, are both associated with the j^{th} level of the spatio-temporal hierarchy. This level is the same for temporal and spatial partial probability distributions, introducing the spatio-temporal coupling in (2). Herein, we marked the calibration parameters by τ_0 and v_0 .

As it is seen from (2), our WM-CTRW formalism belongs to the *non-separable* category of CTRW, which makes the diffusion phase diagram quite rich [16, 17].

From (2), it is easy to derive useful temporal and spatial single-step moments

$$\langle t \rangle = \int_0^\infty dt \, t \int_{-\infty}^\infty dx \, \psi(x, t) = \tau_0 \begin{cases} \frac{1-1/N}{1-\tau/N}, & \text{for } \alpha > 1, \\ \infty, & \text{for } \alpha < 1 \end{cases} \quad (7)$$

where $\alpha = \ln N / \ln \tau$ and

$$\langle x^2 \rangle = \int_{-\infty}^\infty dx \, x^2 \int_0^\infty dt \, \psi(x, t) = (b_0)^2 \begin{cases} 2 \frac{1-1/N}{1-b^2/N}, & \text{for } \beta > 2, \\ \infty, & \text{for } \beta < 2 \end{cases} \quad (8)$$

³ Obviously, detailed forms of the scaling functions f and h are less important for macroscopic displacement and asymptotic long time, respectively. Here, we used their simple explicit forms to make our calculations easier.

where $b_0 = v_0\tau_0$, $b = v\tau$ and $\beta = \ln N / \ln b$.

Notably, α and β are parameters which control the different phases of diffusion. It was proved in [17] that the (total) fractional diffusion exponent η , which governs the asymptotic time-dependence of the walker multi-step mean-square displacement (cf. Equation (11) in [17]),

$$\langle X^2(\Delta t) \rangle \approx \frac{2 D_{st}}{\Gamma(\eta + 1)} (\Delta t)^\eta, \quad (9)$$

itself depends on α and β (cf. Table 1 and diffusion phase diagram shown in Figure 1 in [17]). We term these parameters *temporal* and *spatial diffusion exponents*⁴, respectively. The quantity D_{st} is the so-called *fractional diffusion coefficient* emphasizing its association with the stationary random walk [17]. It depends on partial (temporal and spatial) diffusion exponents (see also Table 1 in [17]). Notably, the total time is $t_{tot} = \sum t$ and displacement of the walker is $X(t_{tot}) = \sum x$ (where \sum means the sum over successive steps of the walker).

Expressions (2) - (6) enable simulation of a random walk trajectory (schematically shown in Figures 1 and 2) in continuous time. This is because they define the corresponding stochastic dynamics (considered in Section V A).

B. Superdiffusion phase

The WM-CTRW defines a stationary stochastic process valid only for the case where the mean waiting-time, $\langle t \rangle$, is finite, i.e. for the case where the temporal exponent $\alpha > 1$ (cf. Equation (7) as well as Equations (4) and (5) in [17]). Under such conditions, the diffusion exponent $\eta = 2H$, where $0 < H \leq 1$, is the well known Hurst exponent [22]. All our analytical calculations are confined to the case where the (multi-step) mean-square displacement is finite for a finite time and superlinearly increases with time for asymptotically long time. That is, we are confined to the superdiffusion phase where $\eta > 1$. This regime is of interest because dragon-kings could be argued to be least relevant in such phase. Our goal is to demonstrate that dragon-kings make a significant impact even in such a superlinear phase.

The superdiffusion phase is restricted by the following inequalities

$$\frac{1}{2} < \frac{1}{\beta} < \frac{1}{2} + \frac{1}{2\alpha}. \quad (10)$$

For the superdiffusion phase, we can easily derive the VAF, in the absence of a dragon-king, in the form (cf. Equation (13) in [17])

$$C(\Delta t) = \frac{2 D_{st}}{\Gamma(\eta - 1)} \frac{1}{\Delta t^{2-\eta}}, \quad (11)$$

where the fractional diffusion coefficient is given by

$$D_{st} = \frac{1 - \frac{\pi}{N}}{\ln N} \frac{\pi\alpha}{\sin\left(2\pi\alpha\left(\frac{1}{\beta} - \frac{1}{2}\right)\right)} \quad (12)$$

and the fractional diffusion exponent is

$$\eta = 1 + 2\alpha\left(\frac{1}{\beta} - \frac{1}{2}\right). \quad (13)$$

As the fractional diffusion exponent $\eta < 2$, the velocity autocorrelation function, $C(\Delta t)$, given by Expression (11) vanishes for extremely long Δt . In Section IV we prove that the presence of a dragon-king in the time series can lead to violation of this property.

IV. DERIVATION OF FORMULAE FOR $C_d(\Delta t)$

In this section we derive relations of two different types between the estimator of $C_d(\Delta t)$ and the estimators of some VAFs concerning time series in absence of any dragon-king. That is, in Section IV A we consider case (i) while in Section IV B case (ii), where both were already defined in Section II.

⁴ It can be also proved that there exists a simple exponent, which is a function of the *partial diffusion exponents* α and β , that governs the time-dependence of the walker single-step mean-square displacement, $\langle [x(t)]^2 \rangle = \int_{-\infty}^{\infty} dx x^2 \psi(x, t)$, for asymptotic time interval t .

A. $C_d(\Delta t)$ for the case of sustained dragon-king

In this section we assume that

- (1) $dt \leq \Delta t \leq \Delta t_{MAX} \ll t_L, t_R$ and
- (2) $\Delta t_{MAX} \leq t_d$, where Δt_{MAX} is a maximal value of Δt .

Importantly, assumption (2) is violated for case (ii) defined in Section II above and considered below in Section IV B.

We divide the expected value of the product of velocities, $\langle v_1 v_2 \rangle(\Delta t)$, present in Formula (1), into five weighted different components. They are estimated by the following terms

$$\langle v_1 v_2 \rangle(\Delta t) = \sum_{m=1}^{M=5} \langle v_1 v_2 \rangle_m(\Delta t) w_m. \quad (14)$$

It is straightforward to derive each component $\langle v_1 v_2 \rangle_m(\Delta t)$ separately by using the proper moving-average estimator. The corresponding weight w_m is easy to obtain.

- (1) The first component (for $m = 1$) is defined for the case, where both velocities v_1 and v_2 are placed before the dragon-king position. This component, together with the corresponding weight are, as follows

$$\langle v_1 v_2 \rangle_1(\Delta t) \stackrel{\text{def.}}{=} \frac{1}{t_L - \Delta t} \sum_{t'=dt}^{t_L - \Delta t} v(t') v(t' + \Delta t), \quad w_1 \stackrel{\text{def.}}{=} \frac{t_L - \Delta t}{t_{tot} - \Delta t} \quad (15)$$

and relates to a random walk in the absence of any dragon-king. More precisely, this component represents the situation where velocity v_1 is placed inside the time interval $[0, t_L - \Delta t]$ while velocity v_2 can be placed both inside this time interval or at its right border $t_L - \Delta t$.

- (2) For the second component ($m = 2$), velocity v_1 is placed before the position of the dragon-king while velocity v_2 equals the velocity of the dragon-king v_d . The second component and the corresponding weight takes the form

$$\langle v_1 v_2 \rangle_2(\Delta t) \stackrel{\text{def.}}{=} \frac{1}{\Delta t} \sum_{t'=t_L - \Delta t + dt}^{t_L} v(t') v_d = \langle v \rangle_L v_d, \quad w_2 \stackrel{\text{def.}}{=} \frac{\Delta t}{t_{tot} - \Delta t}, \quad (16)$$

where v_d is the velocity of the dragon-king (or the slope of the longest straight line shown in Figure 1). This term describes the first cross situation where velocity v_1 is located inside the time interval $[t_L - \Delta t, t_L]$ or at its right border while velocity v_2 is placed inside the time-interval $[t_L, t_L + \Delta t]$ or at its right border. Obviously, v_d is the constant velocity of the dragon-king.

- (3) This case is particularly simple as both velocities v_1 and v_2 are equal to the dragon-king's velocity v_d . The third component ($m = 3$) is the simplest one and together with the corresponding weight assume the forms

$$\langle v_1 v_2 \rangle_3(\Delta t) \stackrel{\text{def.}}{=} v_d^2, \quad w_3 \stackrel{\text{def.}}{=} \frac{t_d - \Delta t}{t_{tot} - \Delta t}, \quad (17)$$

and corresponds to the case where both velocities v_1 and v_2 are placed inside the dragon-king's time interval $[t_L, t_L + t_d]$ or velocity v_2 can be also located at its right border.

- (4) Herein, velocity v_1 equals the dragon-king's velocity v_d while velocity v_2 is placed after the position of the dragon-king. The fourth component ($m = 4$) and the weight are

$$\langle v_1 v_2 \rangle_4(\Delta t) \stackrel{\text{def.}}{=} \frac{1}{\Delta t} \sum_{t'=t_L + t_d - \Delta t + dt}^{t_L + t_d} v_d v(t') = v_d \langle v \rangle_R, \quad w_4 \stackrel{\text{def.}}{=} \frac{\Delta t}{t_{tot} - \Delta t}, \quad (18)$$

corresponding to the second cross case. Precisely, for this case velocity v_1 is placed inside the dragon-king time interval $[t_L + t_d - \Delta t, t_L + t_d]$ or at its right border while velocity v_2 is placed inside the time interval $[t_L + t_d, t_L + t_d + \Delta t]$ or at its right border.

- (5) For the fifth component ($m = 5$), positions of both velocity v_1 and v_2 are placed after the position of the dragon-king. The fifth component and the weight are, as follows

$$\langle v_1 v_2 \rangle_5 (\Delta t) \stackrel{\text{def.}}{=} \frac{1}{t_R - \Delta t} \sum_{t'=t_{tot}-t_R+\Delta t}^{t_{tot}-\Delta t} v(t') v(t' + \Delta t), \quad w_5 \stackrel{\text{def.}}{=} \frac{t_R - \Delta t}{t_{tot} - \Delta t}, \quad (19)$$

being analogous to those of the first component and weight. More precisely, they are defined for the case where both velocities v_1 and v_2 are placed inside the time interval $[t_{tot} - t_R, t_{tot}]$ (without of the dragon-king) and velocity v_2 can be also counted at time t_{tot} .

Subsequently, we can calculate the estimator, which already includes the full influence of the sustained dragon-king

$$\begin{aligned} C_d(\Delta t) &= \langle v_1 v_2 \rangle (\Delta t) - \langle v_1 \rangle \langle v_2 \rangle \\ &= \frac{\gamma_L - \Delta t/t_{tot}}{1 - \Delta t/t_{tot}} \langle v_1 v_2 \rangle_1 (\Delta t) + \frac{\gamma_R - \Delta t/t_{tot}}{1 - \Delta t/t_{tot}} \langle v_1 v_2 \rangle_5 (\Delta t) + \frac{\Delta t/t_{tot}}{1 - \Delta t/t_{tot}} (\langle v \rangle_L + \langle v \rangle_R) v_d + \frac{\gamma_d - \Delta t/t_{tot}}{1 - \Delta t/t_{tot}} v_d^2 \\ &\quad - \frac{1}{(1 - \Delta t/t_{tot})^2} [\gamma_L \langle v \rangle_L + (\gamma_R - \Delta t/t_{tot}) \langle v \rangle_R + \gamma_d v_d] [(\gamma_L - \Delta t/t_{tot}) \langle v \rangle_L + \gamma_R \langle v \rangle_R + \gamma_d v_d] \end{aligned} \quad (20)$$

by using definitions

$$\begin{aligned} \langle v_1 \rangle &\stackrel{\text{def.}}{=} \frac{1}{t_{tot} - \Delta t} \left[\sum_{t'=dt}^{t_L} v(t') + t_d v_d + \sum_{t'=t_{tot}-t_R+\Delta t}^{t_{tot}-\Delta t} v(t') \right] = \frac{1}{t_{tot} - \Delta t} [t_L \langle v \rangle_L + (t_R - \Delta t) \langle v \rangle_R + t_d v_d] \\ \langle v_2 \rangle &\stackrel{\text{def.}}{=} \frac{1}{t_{tot} - \Delta t} \left[\sum_{t'=\Delta t+dt}^{t_L} v(t') + t_d v_d + \sum_{t'=t_{tot}-t_R+\Delta t}^{t_{tot}} v(t') \right] = \frac{1}{t_{tot} - \Delta t} [(t_L - \Delta t) \langle v \rangle_L + t_R \langle v \rangle_R + t_d v_d], \end{aligned} \quad (21)$$

where $\langle v \rangle_L$ and $\langle v \rangle_R$ are partial mean velocities defined by the random walk on the left and right hand side of the dragon-king, respectively (cf. Figure 1). We set velocities $\langle v \rangle_L$ and $\langle v \rangle_R$ equal to zero as no drift is present in the system. Besides, we used dimensionless parameters $\gamma_L \stackrel{\text{def.}}{=} t_L/t_{tot}$, $\gamma_R \stackrel{\text{def.}}{=} t_R/t_{tot}$ and $\gamma_d \stackrel{\text{def.}}{=} t_d/t_{tot}$.

Because drift is absent in the system, Equation (20), taking into account Definitions (21), assumes a simpler form

$$\begin{aligned} C_d(\Delta t) &= \frac{\gamma_L - \Delta t/t_{tot}}{1 - \Delta t/t_{tot}} \langle v_1 v_2 \rangle_1 (\Delta t) + \frac{\gamma_R - \Delta t/t_{tot}}{1 - \Delta t/t_{tot}} \langle v_1 v_2 \rangle_5 (\Delta t) \\ &\quad + \left[\frac{\gamma_d}{1 - \Delta t/t_{tot}} \left(1 - \frac{\gamma_d}{1 - \Delta t/t_{tot}} \right) - \frac{\Delta t/t_{tot}}{1 - \Delta t/t_{tot}} \right] v_d^2. \end{aligned} \quad (22)$$

Apparently, the quantity $C_d(\Delta t)$ depends (in general) not only on the relative variable $\Delta t/t_{tot}$ but also on the parameters γ_L , γ_R , and γ_d . Hence, this quantity depends on the position of the dragon-king inside a time series, i.e. this quantity is, in general, a non-stationary one. However, it does not depend on the sign of the dragon-king velocity. Moreover, the origin of the dragon-king is irrelevant. That is, our derived formula is sufficiently generic in the sense that it is valid not only for the WM-CTRW but for any random walk.

For sufficiently wide time window Δt_{MAX} which still obeys $\Delta t_{MAX}/t_{tot} \ll 1$, the quantity $C_d(\Delta t)$ simplifies into the asymptotic formula

$$C_d(\Delta t) \approx \left[\frac{\gamma_d}{1 - \Delta t/t_{tot}} \left(1 - \frac{\gamma_d}{1 - \Delta t/t_{tot}} \right) - \frac{\Delta t/t_{tot}}{1 - \Delta t/t_{tot}} \right] v_d^2. \quad (23)$$

As long as the dragon-king is present, $C_d(\Delta t)$ does not vanish, even though estimators $\langle v_1 v_2 \rangle_1 (\Delta t)$ and $\langle v_1 v_2 \rangle_5 (\Delta t)$ (which can strongly fluctuate) are decaying.

Obviously, Formula (23) takes a simpler form

$$C_d(\Delta t) \approx \gamma_d (1 - \gamma_d) v_d^2 \quad (24)$$

if a strong but reasonable inequality

$$\frac{\Delta t_{MAX}}{t_{tot}} \ll \min(\gamma_L, \gamma_R, \gamma_d (1 - \gamma_d)) \quad (25)$$

is obeyed.

1. *Important case of $\langle v_1 v_2 \rangle_1(\Delta t) = \langle v_1 v_2 \rangle_5(\Delta t)$*

By assuming that random walks before and after the dragon-king appearance within the time series are statistically identical (although corresponding trajectories could be quite different) we can write, by neglecting unavoidable fluctuations, that $\langle v_1 v_2 \rangle_1(\Delta t) = \langle v_1 v_2 \rangle_5(\Delta t) = C(\Delta t)$. These equalities enable transforming Expression (22) into a stationary form

$$C_d(\Delta t) = \frac{1 - \gamma_d - 2\Delta t/t_{tot}}{1 - \Delta t/t_{tot}} C(\Delta t) + \left[\frac{\gamma_d}{1 - \Delta t/t_{tot}} \left(1 - \frac{\gamma_d}{1 - \Delta t/t_{tot}} \right) - \frac{\Delta t/t_{tot}}{1 - \Delta t/t_{tot}} \right] v_d^2, \quad (26)$$

which is our reference formula. This stationary form is our achievement which enables several applications of Formula (26).

The main difference between $C(\Delta t)$ and $C_d(\Delta t)$ is that the former asymptotically vanishes while the latter does not. This difference provides a tool which allows for distinguishing a power-law relaxation, controlled by rare extremes or black swans, from a decay controlled by the dragon-king. Indeed, we consider Expression (26) as a reference one, also relevant for more complex cases.

If Assumption (25) is valid, further simplification of Expression (26) can be made

$$C_d(\Delta t) \approx (1 - \gamma_d) [C(\Delta t) + \gamma_d v_d^2]. \quad (27)$$

This expression depends on two parameters γ_d and v_d fully characterizing the dragon-king. These parameters can be easily determined, e.g. from the initial and asymptotic nonvanishing values of $C_d(\Delta t)$. In fact, predictions of this simple formula is compared in Section V with corresponding results of simulation.

B. $C_d(\Delta t)$ for the case of a shock (case (ii) of Section II)

To consider the case of a shock within the time-series, we replace Assumption (2), given in Section IV A, by the following one

$$t_d = dt. \quad (28)$$

That is, the duration time of the dragon-king is assumed to be as short as possible and equals to the time discretization step dt . The Assumption (28) imposes a modification of components $m = 2$ and $m = 4$ in Expression (14). Namely, components $m = 2$ and $m = 4$ together with their corresponding weights are replaced by expressions

$$\langle v_1 v_2 \rangle_2 \stackrel{\text{def.}}{=} v_L v_d, \quad w_1 = \frac{dt}{t_{tot} - \Delta t},$$

and

$$\langle v_1 v_2 \rangle_4 \stackrel{\text{def.}}{=} v_d v_R, \quad w_2 = \frac{dt}{t_{tot} - \Delta t},$$

respectively. Furthermore, instead of component $m = 3$ we have

$$\langle v_1 v_2 \rangle_3, \quad w_3 = \frac{\Delta t - dt}{t_{tot} - \Delta t}, \quad (29)$$

where the current velocities $v_1 = v_L$ and $v_2 = v_R$ are now located before and after the position of the shock, respectively. Hence, a new relation for $C_d(\Delta t)$ has the form

$$\begin{aligned} C_d(\Delta t) = & [(t_L - \Delta t) \langle v_1 v_2 \rangle_1(\Delta t) + dt v_L v_d + (\Delta t - dt) \langle v_1 v_2 \rangle_3(\Delta t) + dt v_d v_R + (t_R - \Delta t) \langle v_1 v_2 \rangle_5(\Delta t)] \frac{1}{t_{tot} - \Delta t} \\ & - [t_L \langle v \rangle_L + dt v_d + (t_R - \Delta t) \langle v \rangle_R] [(t_L - \Delta t) \langle v \rangle_L + dt v_d + t_R \langle v \rangle_R] \frac{1}{(t_{tot} - \Delta t)^2}. \end{aligned} \quad (30)$$

By analogy to case (i) of a sustained dragon-king, we set $\langle v_1 v_2 \rangle_1(\Delta t) = \langle v_1 v_2 \rangle_3(\Delta t) = \langle v_1 v_2 \rangle_5(\Delta t) = C(\Delta t)$ that, in the case of no drift present in the system, simplifies (30) into the form

$$C_d(\Delta t) = \frac{t_{tot} - \Delta t - 2dt}{t_{tot} - \Delta t} C(\Delta t) + \frac{(v_L + v_R) X_d}{t_{tot} - \Delta t} - \left(\frac{X_d}{t_{tot} - \Delta t} \right)^2, \quad (31)$$

where $X_d = dt v_d$ is the value of the shock. Note that v_L and v_R are velocities of the walker, which are separated by the time interval $2\Delta t$. From definition of the time step dt we have $dt \ll t_{tot} - \Delta t$. Hence, Equation (31) again assumes a simpler form

$$C_d(\Delta t) = C(\Delta t) + \frac{(v_L + v_R) X_d}{t_{tot} - \Delta t} - \left(\frac{X_d}{t_{tot} - \Delta t} \right)^2, \quad (32)$$

which is our second basic formula. This formula is further modified in Section V to make $C_d(\Delta t)$ better suited for comparison with our results obtained by simulations.

V. ALGORITHM AND RESULTS

A. Stochastic dynamics

The stochastic dynamics or algorithm simulating successive single-step displacements of the walker following a WM-CTRW consists of three stages.

- (i) The drawing of the level index j from distribution (3) in each spatio-temporal step separately.
- (ii) The calculation of the duration time t of the single-step (or its elapsed time) from the stochastic equation

$$t = -\tau_0 \tau^j \ln(1 - R), \quad (33)$$

where $R \in [0, 1]$ is a random number drawn from the random number generator (or from the uniform distribution confined to a unit interval); note that Equation (33) is equivalent to Equation (6) (after an application of well known method of the cumulative distribution function inversion).

- (iii) The determination of the single-step displacement by Equation

$$x(t) = \xi v_0 v^j t, \quad (34)$$

where stochastic variable ξ is a dichotomic noise (i.e. $\xi = +1$ or -1 with equal probability $1/2$) and $x(t) \stackrel{\text{def}}{=} X(t') - X(t' - t)$, where t' is a current time (and not a time interval).

In this way construction of a single continuous in time trajectory of a random walk is possible. Obviously, the parameters τ_0, τ, v_0 , and v were fixed at the beginning of a whole simulation. Further in the text and in all our simulations we set the calibrating parameters $\tau_0 = 1$ and $v_0 = 1$. In fact, this trajectory is constructed in the frame of a non-stationarized version of a CTRW, where there is no special treatment of the initial step [16, 17, 20]. That is, the hierarchical spatio-temporal waiting-time distribution (2) for the initial step is the same as for all other steps. Indeed, the stationarized version of the CTRW, which differently treats the waiting-time distribution for the initial step, corresponds to the WM-CTRW formalism⁵ [17]. The VAF obtained theoretically within the WM-CTRW enables for comparison with the corresponding VAF obtained from simulations. This is possible because in simulations we deal with moving-averages, which by definition, average over the initial state, thus supplying the required asymptotic stationary VAF.

The above given algorithm was used in Section VB to simulate the required basic random walk trajectory. The trajectory constructed in such a way is punctuated "manually" by the sustained dragon-king or by the shock dragon-king. However, preparation of the former dragon-king requires some explanation.

1. Preparation of sustained dragon-king

The stochastic dynamics defined by Equations (33) and (34) is controlled by three random variables: j , R , and ξ , while only the integer level index j , drawn from the distribution (3), and random number R are responsible for the

⁵ It can be proved that the WM-CTRW formalism, even asymptotically, is more general than the fractional Brownian motion introduced by Mandelbrot and van Ness [22]. It is because the propagator derived within the WM-CTRW formalism can be non-Gaussian (for $1/\beta > 1/\alpha$).

size of a single step. For the case of the sustained dragon-king this dynamics is simplified by replacing Equation (33), which simulates an exponential distribution of interevent times (6), by the corresponding reduced expression

$$t = t_d = \tau_0 \tau^{j_d}. \quad (35)$$

That is, the size of the sustained super-extreme event is controlled only by a single random variable $j = j_d$ as are the velocity of the sustained dragon-king $v_d = v_0 v^{j_d}$, its duration time $t_d = \tau_0 \tau^{j_d}$, and displacement $x_d = \xi b_0 b^{j_d}$, where $b_0 = v_0 \tau_0$ and $b = v \tau$.

We can say that the stochastic process defined by Equations (35) and (34) is simplified. This process is a discrete in time and hence in space, where

$$x(t) = X_d = \xi v_0 \tau_0 v^{j_d} \tau^{j_d} = \xi b_0 b^{j_d}. \quad (36)$$

If the index j_d of a given dragon-king is fixed, the interevent times for this process cannot fluctuate. Its discrete-time step equals to the mean-time defining the exponential distribution (33) or (6).

Apparently, we deal with two stochastic processes: (i) the first one which prepares the WM-CTRW trajectory or stochastic spatio-temporal hierarchy of events and (ii) the second process which generates the sustained dragon-king from the simplified, discrete in space and time hierarchical random walk. In fact, the latter process is used in Section V A 2 to illustrate the definition of black swan.

Right now, it is easy to separate in simulation the sustained dragon-king from the rest of events belonging to the stochastic spatio-temporal hierarchy of events (cf. Section V A 2). Namely, it is sufficient to choose the level index j_d much larger than the maximal level index $j = j_{MAX}$ of the hierarchy defining the longest single-step displacement of any other event. The above described simplicity is the main reason for such a way of selection of a sustained dragon-king.

2. Hierarchical random walk and black swans

It is decisive for our considerations that the ratio of successive weights

$$\frac{w(j+1)}{w(j)} = \frac{1}{N}, \quad (37)$$

is already j independent. This means that steps defined by the level index j are N times more likely than those of the next step of larger order $j+1$. Therefore, one expects (on the average) that the walker will perform N^j shorter steps before performing the next step of larger order. Hence, we can explain how extreme events or black swans control the hierarchical spatio-temporal structure of events in the frame of a simplified hierarchical random walk.

Here we consider, as a typical quantity, the mean-square displacement of the process

$$\langle X^2 \rangle(L) = \left\langle \left(\sum_{l=1}^L x_l \right)^2 \right\rangle = L \langle x^2 \rangle, \quad (38)$$

where $\langle \dots \rangle$ denotes an ensemble average, L is the total number of the random walk steps, x_l is a single-step displacement and $\langle x^2 \rangle$ is its mean-square value. In this derivation we neglected the off-diagonal term $\sum_{l \neq l'}^L \langle x_l x_{l'} \rangle$ or crossed correlations between successive single-step displacements in comparison with the diagonal term $L \langle x^2 \rangle$. This is because of the process definition, which invokes independent draw of steps.

In Figure 3, the schematic illustration of the above considerations is shown by using a part of the trajectory or random walk realization consisting of hierarchically ordered steps $(\tau_0 \tau^j, b_0 b^j)$, for $j = 0, 1, 2$. Herein, we neglected (i) fluctuation of the number of hierarchy levels j s as well as (ii) their random succession. Thus we plotted the ordered trajectory within the time-space frame of coordinates. In fact, we made a transformation from the stochastic hierarchy to its deterministic counterpart. The explanation of the concept of black swans becomes now more convenient.

We can easily derive the useful relation between the single-step mean-square displacement $\langle x^2 \rangle$ of the simulated trajectory and the maximal level, j_{MAX} , of the hierarchy contained in it. Indeed, the level j_{MAX} defines the extreme event or black swan by the pair of components $(\tau_0 \tau^{j_{MAX}}, b_0 b^{j_{MAX}})$. This quantity, for a large number of steps $L \gg 1$ or $j_{MAX} \gg 1$, is given by

$$\langle x^2 \rangle \approx (b_0)^2 \left(\frac{N^{j_{MAX}}}{L} (b^2)^0 + \frac{N^{j_{MAX}-1}}{L} (b^2)^1 + \frac{N^{j_{MAX}-2}}{L} (b^2)^2 + \dots + \frac{N^0}{L} (b^2)^{j_{MAX}} \right) \quad (39)$$

where now $\langle \dots \rangle$ means an average over the random walk steps. That is, we used here a kind of an ergodic hypothesis.

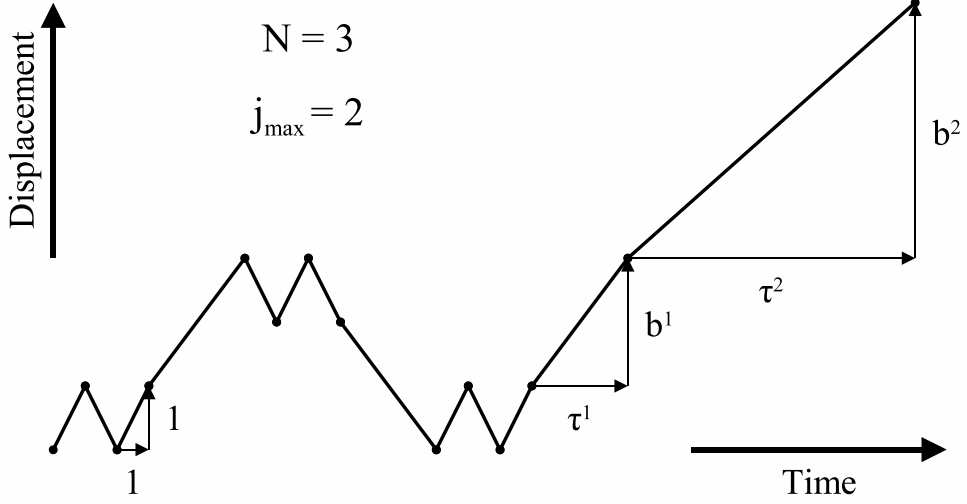


FIG. 3: Schematic trajectory of hierarchically ordered steps (τ^j, b^j) presented, for simplicity, for $N = 3$, $j_{MAX} = 2$ and calibration parameters $\tau_0 = 1$, $b_0 = 1$. Obviously, in our calculations we assumed $j_{MAX} \gg 1$. Herein, the extreme event or black swan is defined by the pair of components $(\tau^{j_{MAX}=2}, b^{j_{MAX}=2})$ directed by $j_{MAX} = 2$.

From Eqs. (38) and (39) we obtain

$$\langle X^2 \rangle(L) \approx (b_0)^2 N^{j_{MAX}} \frac{(b^2/N)^{j_{MAX}+1} - 1}{b^2/N - 1} \approx (b_0)^2 \begin{cases} \frac{1}{1-b^2/N} (b^2)^{j_{MAX}}, & \text{for } b^2/N > 1 \\ \frac{1}{1-b^2/N} N^{j_{MAX}}, & \text{for } b^2/N < 1 \end{cases} \quad (40)$$

or

$$\langle X^2 \rangle(L) \approx (b_0)^2 \begin{cases} \frac{(1-1/N)^{2/\beta}}{1-1/N^{2/\beta-1}} L^{2/\beta}, & \text{for } \beta < 2 \\ \frac{1-1/N}{1-b^2/N} L, & \text{for } \beta > 2 \end{cases} \quad (41)$$

as well as

$$\langle X^2 \rangle(L) \approx \begin{cases} \frac{1}{1-1/N^{2/\beta-1}} (x_{MAX})^2, & \text{for } \beta < 2 \\ \frac{1-1/N}{1-N^{2/\beta-1}} L, & \text{for } \beta > 2 \end{cases} \quad (42)$$

where the number of steps

$$L \approx N^{j_{MAX}} + N^{j_{MAX}-1} + N^{j_{MAX}-2} + \dots + N^0 \approx \frac{1}{1-1/N} N^{j_{MAX}} \quad (43)$$

and the single-step displacement of black swan is defined by

$$x_{MAX} = b_0 b^{j_{MAX}}. \quad (44)$$

Herein, the marginal case of $\beta = 2$ is not considered. Apparently, for $\beta < 2$, i.e. if $\langle X^2 \rangle$ scales with L according to some power-law, $\langle X^2 \rangle$ is fully determined by x_{MAX}^2 . This is exactly what we need for illustration of our considerations. That is, the quantities (herein, the mean-square displacement of the process) which characterize the system are mainly expressed by the corresponding ones (herein, a single-step displacement) which define the black swans.

Note that in the case of the WM-CTRW we have results analogous to those presented by Equations (41) and (42). However, their derivation is much more complicated in this case. The threshold property exhibited by these equations is typical of the behavior of other key quantities, like volatilities or correlation functions.

As expected, for Brownian motions (that is for the case of $\beta > 2$ in Equations (40), (41) and (42)), the factor preceding L in the second formula in (41) equals, in fact, to the corresponding factor present in Formula (8). Here, the absence of factor 2 is only caused by the absence of fluctuations of interevent times.

Now, we are ready to answer the question concerning the distribution of the single-step displacements $|x| = b_0 b^j$ of the walker. This answer is based on the change of variables from j to $|x|$. Hence, the corresponding (normalized) distribution $\tilde{w}(|x|)$ takes the Pareto form

$$\tilde{w}(|x|) \approx \frac{1}{b_0} \frac{\beta}{(|x|/b_0)^{\beta+1}}, \quad (45)$$

where $|x| \geq b_0$. In fact, Equation (45) holds only for $|x| \gg b_0$. It is well known [12, 21] that the power-law distribution of events leads to the Fréchet distribution of extreme events. For asymptotic values of the argument, this distribution preserves the power-law of exponent $\beta + 1$ with the power-law correction to scaling of exponent β . Note that Equation (45) is valid for all values of β . That is, black swans are always present in the system. However, only for $\beta < 2$ their influence dominates.

To conclude this discussion, we can say that the linear size of the walk (herein, the mean-square displacement) is controlled by extreme events or black swans if this size scales with the number of steps according to some power-law, i.e. if the walk is a kind of fractional random walk. Otherwise, the Brownian motions dominate and no influence of black swans is observed (although they are present in the system). That way the threshold functions as a discriminator of black swans.

It is evident now, that we can identify an event as a sustained dragon-king if j_d index of this event distinctly exceeds that of j_{MAX} .

B. Comparison of theoretical predictions with simulations

Herein, we restrict our simulations to an important example of a basic random walk given by a fractional Brownian motion. That is, we study a confined region of a superdiffusion phase defined by $1/\beta$ only slightly smaller than $1/\alpha$. Note that inequality $1/\beta < 1/\alpha$ is equivalent to $v < 1$, i.e. it corresponds to the case where the velocity of the walker for the higher hierarchy level is smaller. In this case, each moment of the arbitrary non-negative order of the (multi-step) displacement is finite for finite times [17, 18]. This choice of such moments arises from empirical evidences that these moments are always finite. Other choices, concerning other diffusion phases, would also be worth studying. The dragon-king is located "manually" inside the simulated time series in such a way that inequality $\Delta t_{MAX} \ll t_L, t_R$ is obeyed.

1. Results for sustained dragon-king

Now, we discuss the case where the multiplicative factor preceding v_d^2 in Equation (26) is positive, which is easy to fulfil. That is, we consider inequality $\frac{\Delta t}{t_{tot}} < \gamma_d \left(1 - \frac{\gamma_d}{1 - \Delta t/t_{tot}}\right)$. This inequality is only slightly stronger than $\frac{\Delta t}{t_d} < 1$, yet needed for the derivation of any of our expression for $C_d(\Delta t)$ in the case of the appearance of the sustained dragon-king.

In cumulative Table I we present data (in a form of four inverted pyramids of numbers), which define unnormalized statistics, $S(j)$, of hierarchy levels j s (cf. Section V A). These data were obtained for fixed common parameters $\tau_0 = 1.0$, $\tau = 2.520$, $v_0 = 1.0$, $v = 0.992$, and $N = 4$. They were used to prepare four trajectories within the WM-CTRW formalism in the presence of sustained dragon-kings. For instance, the row numbered by level $j = 3$ gives, at intersection with the second column, the number which says how many times (herein, it is 809231) this level appeared in the first trajectory. This trajectory contains the only sustained dragon-king defined by index $j_d = 13$. This index is shown in Table I by bold number 1, at the intersection of row numbered by level $j = 13$ with the second column, again. The successive columns from three to five contain analogous unnormalized statistics but for an increasing j_d values⁶, i.e. $j_d = 15, 17$, and 19 ; that is, for increasing sustained dragon-kings. The common bottom of all hierarchies (represented in Table I by inverted pyramids of numbers) is placed at the level $j = j_{MAX} = 12$, that is above any $13 \leq j = j_d \leq 19$.

⁶ Equations (35) and (36) precisely define the role of index j_d used here.

TABLE I: Four unnormalized statistics $S(j)$ of hierarchy levels js obtained for four sustained dragon-kings.

Level j	$S(j)$ for $j_d = 13$	$S(j)$ for $j_d = 15$	$S(j)$ for $j_d = 17$	$S(j)$ for $j_d = 19$
0	51763445	51439530	49410801	36182538
1	12948042	12866869	12360063	9049915
2	3234819	3214452	3087567	2260332
3	809231	804047	772246	565401
4	202591	201289	193303	141499
5	50583	50521	48326	35374
6	12773	12704	12211	8895
7	3162	3141	3012	2212
8	811	801	765	565
9	192	191	181	128
10	48	47	43	29
11	6	6	6	4
12	5	5	4	3
13	1	0	0	0
14	0	0	0	0
15	0	1	0	0
16	0	0	0	0
17	0	0	1	0
18	0	0	0	0
19	0	0	0	1

So, all sustained dragon-kings are marked in Table I by the bold number 1. They are placed from the second to fifth column at intersections with the corresponding rows indexed by levels from $j = j_d = 13$ to $j = j_d = 19$. That is, these levels systematically move away from the common bottom of hierarchies.

Importantly, the left part of trajectory (preceding the dragon-king appearance) is common for all dragon-kings. The right border of this part is fixed defining the beginning of a dragon-king. Because the total duration time, t_{tot} , of all trajectories is the same, the duration time, t_R , of the part of trajectory placed on the right-hand side of the dragon-king decreases as the duration time t_d increases (i.e. when the index j_d increases). Therefore, the statistics $S(j)$ of hierarchy levels js , shown in Table I, decreases as the level defining the sustained dragon-king, j_d , is risen (because then the corresponding trajectory or time series, placed on the right-hand side of the dragon-king, is shorter). Therefore, for example, the number 128 placed at intersection of the fifth column and the row denoted by the index level $j = 9$ is distinctly smaller than the number 192 placed at the same row but at the intersection with the second column. We hope that Table I well illustrates the hierarchical structure of any (long) trajectory simulated within the WM-CTRW formalism. Moreover, the corresponding localisations of the sustained dragon-kings in the space of hierarchy levels relative to the inverted pyramids are also well delineated.

In Figure 4, we compare the prediction of Formula (26) (thin solid curves) with results of simulation (dispersed thick solid curves) for four different values of t_d , namely, $t_d/\Delta t_{MAX} = 1.653, 10.496, 66.651$, and 423.263 , which correspond to $j_d = 13, 15, 17$, and 19 , respectively. Note that the current width of the time-window, Δt , (called also *time lag*) ranges from $\Delta t = dt$ up to $\Delta t = \Delta t_{MAX} = 10^5 dt$ with the time step, $dt = 1$, while $t_{tot} = 1400 \Delta t_{MAX}$ is the same for all statistics $S(j)$ shown in Table I. The sustained dragon-kings' time lags were calculated for the same value of $\tau = 2.520$ and the single-step displacements were calculated for the common $b = v\tau = 2.50$ (where $b_0 = v_0\tau_0 = 1.0$).

The upward convexity of the curves in Figure 4 is due to the presence of the corresponding sustained dragon-king. As we expected, the agreement shown in Figure 4 between the prediction of Formula (26) and data obtained from simulations becomes better the further the dragon-king is located from the top of the hierarchy (see the location of bold number 1 in Table I). The best agreement is obtained for the largest $j_d = 19$. In other words, the dragon-kings defined by j_d smaller than 19 slightly positively deviate their corresponding velocity autocorrelation functions, $C_d(\Delta t)$, from their simulational counterparts.

The quantity $C(\Delta t)$, controlled by black swans and given by Expression (11), is also plotted in Figure 4 (the dashed curve) as a reference VAF, i.e. this VAF was calculated for the absence of a dragon-king.

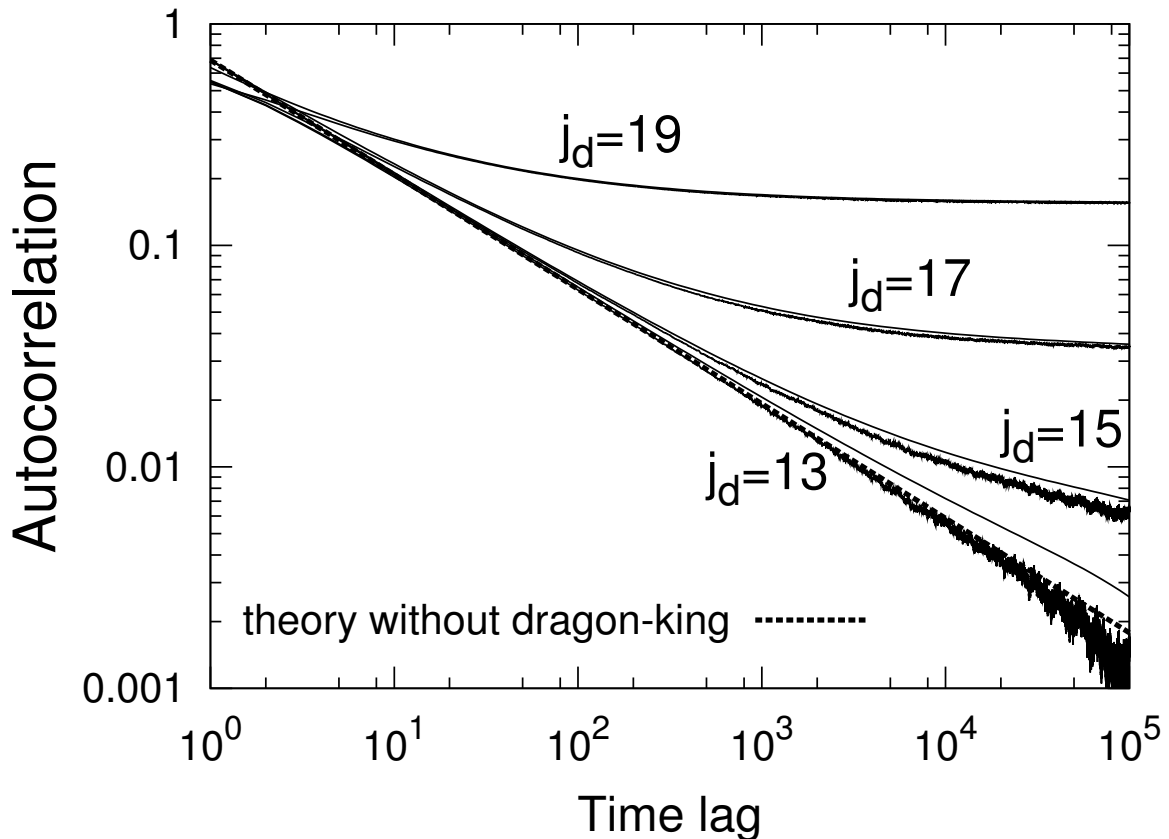


FIG. 4: Comparison of the prediction of Equation (26) (solid thin curves) with results of simulations (dispersed solid thick curves) for four different values of $t_d/\Delta t_{MAX} = 1.653, 10.496, 66.651$, and 423.263 which correspond to $j_d = 13, 15, 17$, and 19 , respectively. The dashed curve represents the prediction of Equation (11), i.e. the prediction for the time series in the absence of a dragon-king. All curves were calculated for the same values of $\tau = 2.52$, $v = 0.992$, and $N = 4$. Notably, theoretical predictions for $j_d = 17$ and 19 are almost indistinguishable (within the resolution of the plot) from results of the corresponding simulations for the whole range of time lag Δt . The presence of the sustained dragon-king twists upward the curve deviating it from the straight line (in the log – log plot).

2. Results for the shock dragon-king

In Figure 5, we present results of simulations of the VAF for three different values of the shock size $X_d = 0.41, 2.44$ and $5.26 [\times 10^6]$ which corresponds to $X_d/x_{MAX} = 1.90, 11.3$, and 24.4 , respectively, where x_{MAX} is the maximal spatial value of the random walk's single step belonging to the simulated hierarchical WM-CTRW trajectory in the absence of a dragon-king. Remarkably, all these shocks are fully exogenous as they were taken from outside of the spatio-temporal structure of time series or random walks. The simulated trajectories have also the total time $t_{tot} = 1400 \Delta t_{MAX}$ and, except for the presence of the dragon-king, all trajectories are identical.

A striking property of simulated VAFs is their dispersion behaving like a certain instability. Therefore, it is more convenient to use a formula that only describes the dispersion of the data. In principle, such a formula could be obtained by replacing the sum of velocities $v_L + v_R$ in Formula (32) by its dispersion $\sigma = \sqrt{\langle (v_L + v_R)^2 \rangle} = \sqrt{2\sqrt{\sigma_v^2 + C(2\Delta t)}}$, where $\sigma_v^2 = \langle v_L^2 \rangle = \langle v_R^2 \rangle$. This is allowed because we assumed, for simplicity, that a shock does not change the type of the random walk.

Moreover, our approach allows us to study a more realistic case, e.g. in which dispersion is smaller than σ defined above. Hence, we propose a more flexible stationary formula

$$C_d(\Delta t) = C(\Delta t) \pm \sqrt{2} \sigma_f \frac{X_d}{t_{tot} - \Delta t} - \left(\frac{X_d}{t_{tot} - \Delta t} \right)^2, \quad (46)$$

where $\sigma_f \stackrel{\text{def.}}{=} \sqrt{f \sigma_v^2 + C(2\Delta t)}$ and the phenomenological factor or weight, $0 < f \leq 1$, is the same for all trajectories.

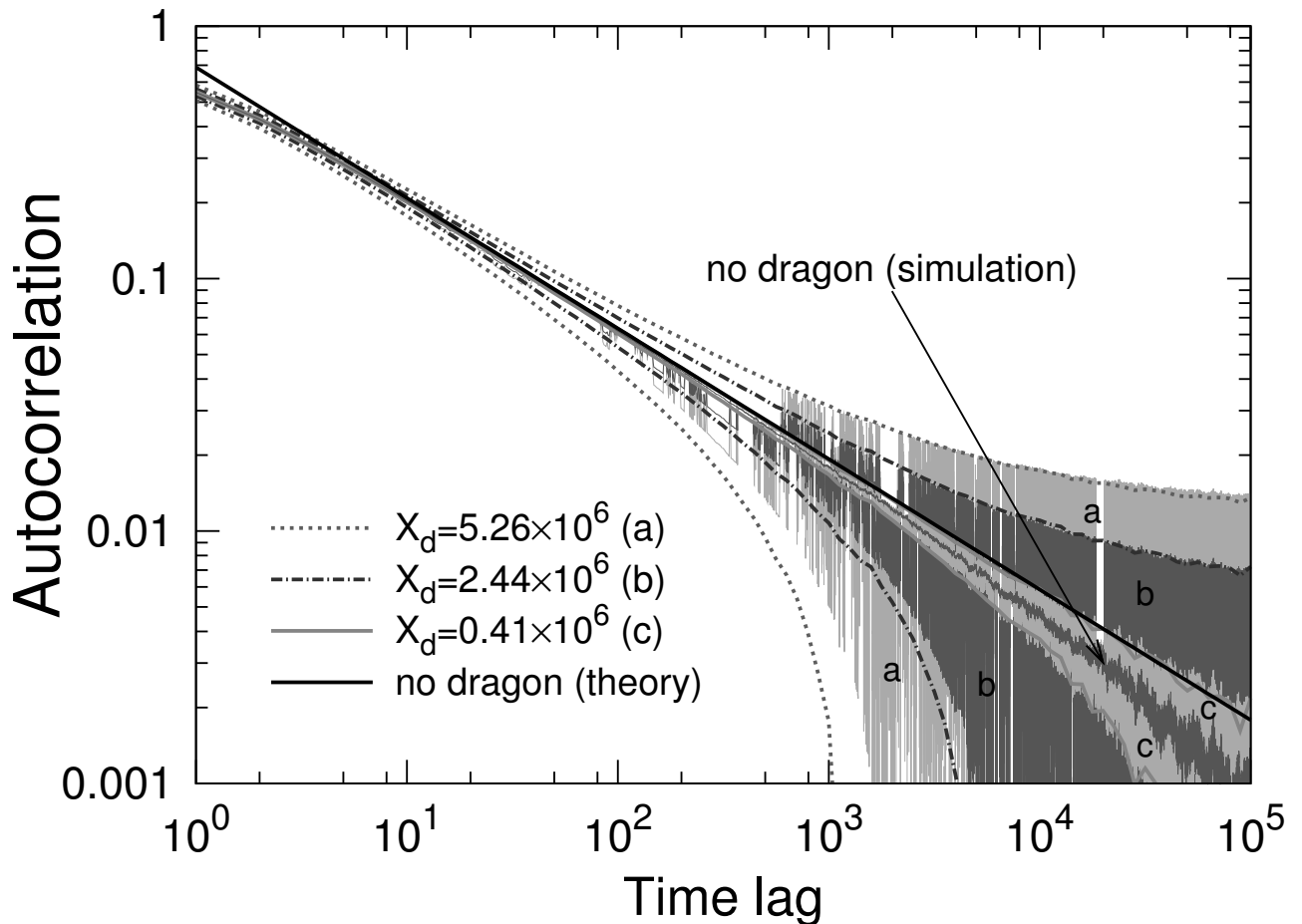


FIG. 5: Comparison of the prediction of Formula (46) (dotted and dashed-dotted curves as well as grey solid one) with results of simulations (corresponding regions having different greyness, additionally marked by a, b and c) for three different values of $X_d = 5.26, 2.44$, and $0.41 [\times 10^6]$. The black solid curve shows the prediction of Formula (11), i.e. the prediction for the time series simulated in the absence of a shock dragon-king; the corresponding simulated VAF is given for this time series by the innermost (dark) region. All curves were obtained for values of $\tau = 2.52$, $v = 0.992$, and $N = 4$, the same as for the case of the sustained dragon-king.

As it is seen, also in the case of the shock dragon-king we transformed the non-stationary Expression (32) to more useful stationary Expression (46).

Comparison of the prediction of Equation (46) with the data obtained by simulation is shown in Figure 5. In this figure, only small deviations are seen for the choice of the factor $f = 0.30$. The data scatter is reasonably small but it increases with the increase of the ratio X_d/t_{tot} . The origin of this scatter comes from fluctuations of the simulated trajectory, unfortunately resulting also in a spontaneous artificial trend (e.g. as a deviation from the power-law in the absence of the dragon-king). Additionally, this trend can be supported by the finite size of the simulated time series.

Note that further simplification of Equation (46) is also possible

$$C_d(\Delta t) = C(\Delta t) \pm \sqrt{2} \sigma_f \frac{X_d}{t_{tot}} - \left(\frac{X_d}{t_{tot}} \right)^2, \quad (47)$$

if the strong but reasonable inequality $\Delta t_{MAX}/t_{tot} \ll 1$ is obeyed.

VI. SUMMARY AND CONCLUDING REMARKS

In the present work we discussed the following issues.

- (i) We considered the influence of two distinctive types of dragon-kings on the velocity autocorrelation function, namely the sustained dragon-king and shock dragon-king.
- (ii) By simulations and by theoretical analysis, we found that the dragon-king influence decisively changes the original VAFs calculated, for instance, within the hierarchical Weierstrass-Mandelbrot Continuous-Time Random Walk formalism for a wide range of time intervals. The influence of both types of dragon-kings is well pronounced but quite different (cf. the corresponding plots shown in Figures 4 and 5). Remarkably, the results obtained by simulations agree well with the corresponding predictions of our simple theoretical Formulas (26) and (46) (see again Figures 4 and 5).
- (iii) Furthermore, several intermediate formulas, e.g. (20), (22), or (30), derived in this work for $C_d(\Delta t)$, can be applied to more complex cases where, for instance, (a) after the dragon-king appearance the random walk is changed, (b) the random walk with drift is considered and (c) dragon-kings cluster in the system.

As it is apparent from Figure 4, the presence of the sustained dragon-king throws the system away from the state controlled by black swans. That is, the 'finger-print' of this dragon-king is significant because it convexly twists upward the autocorrelation curve dependence on the time lag and distinctly deviates it positively from the power-law generated by black swans (cf. Section V A 2 for details). This deviation is one of the main features which distinguishes $C_d(\Delta t)$ from usual $C(\Delta t)$. This difference allows one to distinguish a power-law relaxation, controlled by black swans, from the decay controlled by the sustained dragon-king. Moreover, this difference can be so distinct that it can effectively be applied as a tool to detect the sustained dragon-king from an empirical time series.

The plot in Figure 5 shows that the shock dragon-king also significantly changes $C(\Delta t)$. However, this change is different than the change caused by the sustained dragon-king. This change is also well seen by direct comparison of Formulas (26) and (46). Noticeably, the scatter of the data shown in Fig. 5 indicates some instability of the system after the appearance of the shock dragon-king. Indeed, if the empirical VAF reveals such an instability then we can anticipate that the corresponding empirical time series contains the shock dragon-king. That is, such an anomalous VAF indicates shocks.

Acknowledgments

We thank Armin Bunde for stimulating discussion. This work was partially supported by the Grant No. 119 awarded within the First Competition of the Committee of Economic Research, organized by the National Bank of Poland.

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